Available online at www.sciencedirect.com

SCIENCE DIRECT®

Discrete Applied Mathematics 154 (2006) 1019–1022

DISCRETE APPLIED MATHEMATICS

www.elsevier.com/locate/dam

Note

A note on acyclic domination number in graphs of diameter two

T.C. Edwin Cheng[a], Yaojun Chen[a, b], C.T. Ng[a]

[a]Department of Logistics, The Hong Kong Polytechnic University, Hung Kom, Kowloon, Hong Kong, P.R. China
[b]Department of Mathematics, Nanjing University, Nanjing 210093, PR China

Received 11 December 2003; received in revised form 27 April 2005; accepted 12 September 2005
Available online 28 November 2005

Abstract

A subset S of the vertex set of a graph G is called *acyclic* if the subgraph it induces in G contains no cycles. S is called an *acyclic dominating* set of G if it is both acyclic and dominating. The minimum cardinality of an acyclic dominating set, denoted by $\gamma_a(G)$, is called the *acyclic domination number* of G. Hedetniemi et al. [Acyclic domination, Discrete Math. 222 (2000) 151–165] introduced the concept of acyclic domination and posed the following open problem: if $\delta(G)$ is the minimum degree of G, is $\gamma_a(G) \leqslant \delta(G)$ for any graph whose diameter is two? In this paper, we provide a negative answer to this question by showing that for any positive k, there is a graph G with diameter two such that $\gamma_a(G) - \delta(G) \geqslant k$.

Keywords: Acyclic domination number; Diameter two

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph without loops. The neighborhood $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The minimum degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. If $G[S]$ contains no edge, then we call S an independent set. The *distance* of two distinct vertices u and v, denoted by $d(u, v)$, is the length of a shortest path connecting u and v. The *diameter* of G, denoted by $diam(G)$, is defined as:

$$diam(G) = \max\{d(u, v) \mid u, v \in V(G)\}.$$

A set $S \subseteq V(G)$ is called a dominating set if every vertex u in $V(G) - S$ is adjacent to at least one vertex v in S. For $X, Y \subseteq V(G)$, we say X dominates Y (or Y is dominated by X) if $N(y) \cap X \neq \emptyset$ for any vertex $y \in Y$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A set $S \subseteq V(G)$ is called an acyclic set if $G[S]$ contains no cycles. A set $S \subseteq V(G)$ is called an *acyclic dominating* set of G if it is both acyclic and dominating. The minimum cardinality of an acyclic dominating set in a graph G is called the *acyclic domination number* of G, denoted by $\gamma_a(G)$.

In [4], one can find an appendix listing 75 different types of domination-related parameters that have been studied in the literature (see for instance [1,2]). The concept of acyclic domination was introduced by Hedetniemi et al. [5]. This invariant is particularly interesting in that it is a fundamental type of domination and lies between $\gamma(G)$ and $i(G)$, the

E-mail address: lgtcheng@polyu.edu.hk (T. Cheng).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2005.09.012

minimum cardinality of an independent dominating set. In the same paper, they posed some open questions on acyclic domination including the following.

Question 1. Let G be a graph with $diam(G) = 2$. Is $\gamma_a(G) \leqslant \delta(G)$?

It is shown in [3] that $\gamma_a(G) \leqslant \delta(G)$ does not hold when $\delta(G) = 3$. In this paper, we show that for any positive integers k and $d \geqslant 3$, there is a graph G of diameter two with $\delta(G) = d$ such that $\gamma_a(G) - \delta(G) \geqslant k$.

2. Construction

Let $m \geqslant 2$ and $n \geqslant 2$ be integers and $H(m, n)$ be a graph of order mn with vertex set and edge set as follows:

- $V(H(m, n)) = \bigcup_{1 \leqslant i \leqslant m} \{a_{ij} \mid 1 \leqslant j \leqslant n\}$;
- $E(H(m, n)) = \left(\bigcup_{1 \leqslant i \leqslant m} \{a_{ij}a_{ik} \mid j \neq k\}\right) \cup \left(\bigcup_{1 \leqslant i \leqslant n} \{a_{ji}a_{ki} \mid j \neq k\}\right)$.

Let F be a complete graph of order $d + 1$ ($d \geqslant 3$) with $V(F) = \{v_i \mid 0 \leqslant i \leqslant d\}$. Take $n = dt$, where t is an integer not less than 2. Let $G(d, n)$ be a graph of order $n^2 + d + 1$ with vertex set and edge set as follows:

- $V(G(d, n)) = V(F) \cup V(H(n, n))$;
- $E(G(d, n)) = E(H(n, n)) \cup E(F) \cup \left(\bigcup_{1 \leqslant k \leqslant d} \{v_k a_{ij} \mid 1 \leqslant i \leqslant n, (k-1)t + 1 \leqslant j \leqslant kt\}\right)$.

From the definition of $H(m, n)$, it is easy to see that $H(m, n)$ is the Cartesian product of two complete graphs K_m and K_n, that is, $H(m, n) = K_m \square K_n$. Thus we can easily obtain the following two lemmas.

Lemma 1. $\gamma(H(m, n)) = \gamma_a(H(m, n)) = \min\{m, n\}$.

Lemma 2. $diam(G(d, n)) = 2$.

Lemma 3. $\gamma_a(G(d, n)) = (d - 2)n/d + 2$.

Proof. Let S be an acyclic dominating set of $G(d, n)$ and $|S \cap N(v_0)| = l$. Obviously, $0 \leqslant l \leqslant 2$. If $l = 0$, then in order to dominate $\{v_0\} \cup V(H(n, n))$, we have $|S| \geqslant n + 1$ by Lemma 1. If $l \neq 0$, we assume $S \cap N(v_0) = \{v_1, \ldots, v_l\}$ and $G' = G(d, n) - \bigcup_{1 \leqslant j \leqslant l} N[v_j]$. It is easy to see that $G' = H(n, (d - l)t)$. In order to dominate $V(G')$, S must contain at least $(d - l)t$ vertices of $V(H(n, n))$. Thus, we have $|S| \geqslant (d - l)t + l$. On the other hand, $\{v_1, \ldots, v_l\} \cup \{a_{ii} \mid lt + 1 \leqslant i \leqslant n\}$ is an acyclic dominating set of order $(d - l)t + l$, and hence we have $\gamma_a(G(d, n)) = \min\{n + 1, (d - 1)t + 1, (d - 2)t + 2\} = (d - 2)t + 2 = (d - 2)n/d + 2$. $\square$

Theorem 1. *For any positive integers k and $d \geqslant 3$, there is a graph G of diameter two with $\delta(G) = d$ such that $\gamma_a(G) - \delta(G) \geqslant k$.*

Proof. Take $G = G(d, n)$. By Lemmas 2 and 3, we have $diam(G) = 2$ and $\gamma_a(G) = (d - 2)n/d + 2$. Obviously, $\delta(G) = d$. Thus, $\gamma_a(G) - \delta(G) = (d - 2)n/d + 2 - d$. Since $(d - 2)n/d + 2 - d \to \infty$ as $n \to \infty$ for a fixed d, it is not difficult to see that the conclusion holds. $\square$

As for the domination number of $G(d, n)$, we have the following result.

Theorem 2. $\gamma(G(d, n)) = d$.

Proof. Let S be a minimum dominating set of $G(d, n)$. Since $N(v_0)$ is a dominating set, we have $|S| \leqslant d$. We now show that $|S| = d$. Suppose to the contrary that $|S| < d$. If $S \cap N(v_0) = \emptyset$, then in order to dominate $\{v_0\} \cup V(H(n, n))$, we have $|S| \geqslant n + 1 \geqslant 2d + 1$ by Lemma 1, a contradiction. Hence we may assume without loss of generality that $S \cap N(v_0) = \{v_{l+1}, v_{l+2}, \ldots, v_d\}$. If $l \geqslant 1$, then we must have $S - N(v_0) \neq \emptyset$ and $S - N(v_0) \subseteq V(H(n, n))$. Let

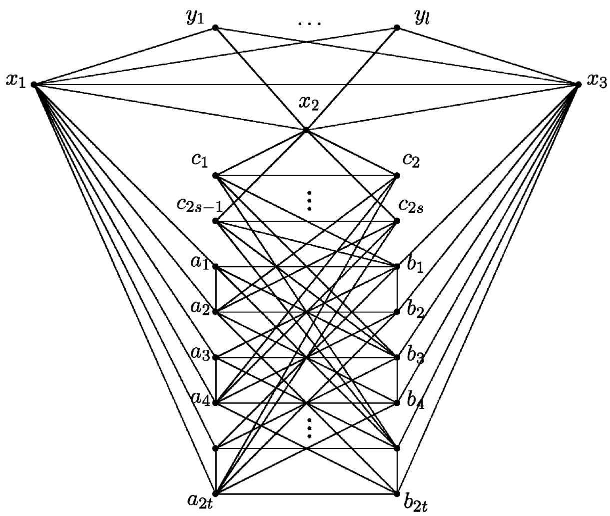

Fig. 1.

$G' = G(d, n) - \bigcup_{l+1 \leqslant j \leqslant d} N[v_j]$, then $G' = H(n, lt)$. In order to dominate $V(G')$, S must contain at least lt vertices of $V(H(n, n))$. Thus, we have $|S| \geqslant (d-l) + lt > d$, a contradiction. Therefore, we have $l = 0$, that is, $S \cap N(v_0) = N(v_0)$. Noting that $N(v_0)$ is a dominating set, we have $\gamma(G(d, n)) = d$. □

Corollary 1. *$N(v_0)$ is the unique minimum dominating set of $G(d, n)$.*

3. Final remark

Let G be a graph. If $diam(G) = k$ and $diam(G - e) > k$ for any edge $e \in E(G)$, then we call G *k-diameter-critical*. It is easy to see that $G(d, n)$ is not 2-diameter-critical since the graph G_0 obtained from $G(d, n)$ by deleting all the edges $v_i v_j$ $(1 \leqslant i < j \leqslant d)$ has diameter two. Since $diam(G_0) = 2$ and G_0 is a subgraph of $G(d, n)$, we have $\gamma(G_0) = d$ by Theorem 2. Obviously, $\gamma_a(G_0) = \gamma(G_0) = \delta(G_0) = d$ if n is large enough. Let $G(l, s, t)$ be a graph as shown in Fig. 1 , where $l \geqslant 1$, $s \geqslant 2$ and $t \geqslant 3$. It has been shown in [3] that $\gamma(G(l, s, t)) = 3$ and $\gamma_a(G(l, s, t)) > \delta(G(l, s, t)) = 3$. It is worth noting that $G(l, s, t)$ is not 2-diameter-critical either. In fact, $G(l, s, t) - x_1x_3$ is a 2-diameter-critical graph and $\gamma(G(l, s, t) - x_1x_3) = \delta(G(l, s, t) - x_1x_3) = 3$. A natural problem is the following.

Question 2. Let G be a 2-diameter-critical graph. Is $\gamma_a(G) \leqslant \delta(G)$?

If the answer to the question above is "YES", then the upper bound for $\gamma_a(G)$ is the best possible in the sense that "$\leqslant$" cannot be replaced by "$<$" as can be seen by the graphs G_0 and $G(l, s, t) - x_1x_3$.

Acknowledgements

We are grateful to the referees for their careful comments on our earlier version of this paper. This research was supported in part by The Hong Kong Polytechnic University under the Grant number G-YX04. The second author was also supported by the National Natural Science Foundation of China under the Grant Number 10201012.

References

[1] R.B. Allan, R. Laskar, On domination and some related concepts in graph theory, Congr. Numer. 21 (1978) 43–58.
[2] B. Bollobás, E.J. Cockayne, Graph-theoretic parameters concerning domination, independence and irredundance, J. Graph Theory 3 (1979) 241–249.

[3] Y.J. Chen, Y.Q. Zhang, K.M. Zhang, Is $\gamma_a \leqslant \delta$ for graphs which have diameter two?, J. Systems Sci. Complexity 16 (2003) 195–198.
[4] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, 1998.
[5] S.M. Hedetniemi, S.T. Hedetniemi, D.F. Rall, Acyclic domination, Discrete Math. 222 (2000) 151–165.